The Mathematics of Beam Deflection

The Royal Academy of Engineering

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C&BE CONSTRUCTION AND THE BUILT ENVIRONMENT DIPLOMA

Scenario

As a structural engineer you are part of a team working on the design of a prestigious new hotel complex in a developing city in the Middle East. It has been decided that the building will be constructed using structural steelwork and, as the design engineer, you will carry out the complex calculations that will ensure that the architect's vision for this new development can be translated into a functional, economic and buildable structure.

As part of these calculations you must assess the maximum deflections that will occur in the beams of the structure and ensure that they are not excessive. In this exercise you will apply numerical integration techniques to solve some typical beam deflection design problems using techniques that form the basis of the calculations that would be undertaken in real life albeit often carried out using sophisticated and powerful computer analysis software.

Importance of Exemplar in Real Life

Structures such as buildings and bridges consist of a number of components such as beams, columns and foundations all of which act together to ensure that the loadings that the structure carries is safely transmitted to the supporting ground below.

Normally, the horizontal beams can be made from steel, timber or reinforced concrete and have a cross sectional shape that can be rectangular, T or I shape. The design of such beams can be complex but is essentially intended to ensure that the beam can safely carry the load it is intended to support. This will include its own self-weight, the weight of the structure it is supporting and what is often referred to as "live load" being the weight of people and furnishings in buildings or the weight of road or rail traffic in bridges.

Examples of beams can be seen in figures 1 to 4

Figure 1: *Steel Beams*

Figure 2: *Bridge Beams*

Figure 3: *Reinforced Concrete Beams*

Figure 4: *Cantilevered Timber Beams*

In addition to the requirements for the beam to safely carry the intended design loads there are other factors that have to be considered including assessing the likely deflection of the beam under load. If beams deflect excessively then this can cause visual distress to the users of the building and can lead to damage of parts of the building including brittle partition dividers between rooms and services such as water and heating pipes and ductwork.

Beam design is carried out according to principles set out in Codes of Practice and typically the maximum deflection is limited to the beam's span length divided by 250. Hence a 5m span beam can deflect as much as 20mm without adverse effect. Thus, in many situations it is necessary to calculate, using numerical methods, the actual beam deflection under the anticipated design load and compare this figure with the allowable value to see if the chosen beam section is adequate.

Background Theory

To calculate beam deflections a standard fundamental formula is used to determine deflections base on beam *curvature.* This is given by the expression:

$$\text{Curvature} = \frac{1}{R} = \frac{M}{EI} = -\frac{d^2v}{dx^2} \qquad \ldots(1)$$

where:

R = The radius of the shape of the curved beam at a distance *x* from the origin, normally taken at the left or right hand end of the beam

E = The Elastic or Young's modulus of the material from which the beam is fabricated. For steel this can be assumed to be 210 kN/mm^2.

I = The *Second Moment of Area* of the beam's cross-section. This value depends on the shape of the cross section and is normally obtained from tables. Its units are m^4 or mm^4 or cm^4. See http://www.rainhamsteel.co.uk/products/universal_beams2nonsfb.html for typical section sizes in structural steelwork

M = The *Bending Moment* at the section, distance *x* from the origin

v = The vertical deflection at the section distance *x* from the origin.

In the above formula *E* and I are normally constant values whilst *v*, x and *M* are variables. *M* can be expressed in terms of distance x and hence integrating the above expression twice will enable the deflection *v* to be calculated.

In other words:

$$\frac{d^2v}{dx^2} = -\frac{M}{EI}$$

$$\frac{dv}{dx} = -\int \frac{M}{EI}dx \qquad \ldots(2)$$

$$v = \int \frac{dv}{dx}dx$$

The *Bending Moment* is a means of describing mathematically the amount of bending and deflection that will occur in a beam under a given loading system and is defined as *the sum of the moments of all forces to the left or right of the section* under consideration. It doesn't matter whether moments are taken to the left or right as the answer will be the same in both cases. For example in figure 5 below the *simply supported* beam shown carries a *uniformly distributed load* of *10 KN/m.* (note the units). When a load is described as *uniformly distributed* it means that the load intensity is the same throughout. The total load on the span will be 5x10 = 50 kN and hence the supporting reactions as marked on the diagram will each be 25 kN.

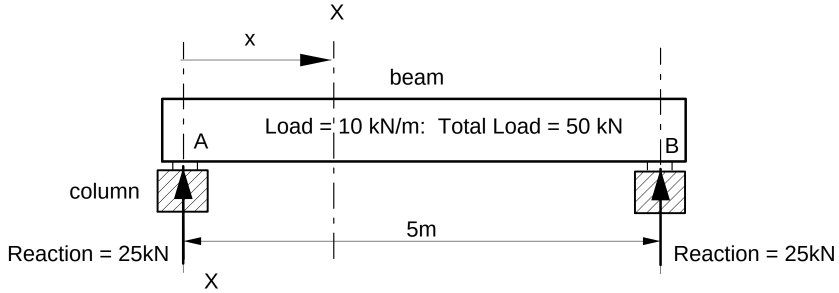

Figure5: *Loads and Reactions on a simply supported beam*

At X, a distance *x* from the left hand support the *Bending Moment, M,* can be calculated by taking moments of all forces to the left of X with the convention that *clockwise moments* are taken as *positive.* Hence, the bending moment at X will be given as:

M = the clockwise moment of the 25kN support reaction – the anticlockwise moment of the *uniformly distributed load*.

i.e. $$M = (25\times x) - (10\times x\times \frac{x}{2}) = 25x - 5x^2 \qquad \ldots (3)$$

Hence combining all the above expressions we can say that:

$$\frac{d^2v}{dx^2} = -\frac{M}{EI} = -\frac{25x - 5x^2}{EI}$$

$$\therefore \quad \frac{dv}{dx} = -\frac{1}{EI}\int(25x - 5x^2)dx = -\frac{1}{EI}\left[\frac{25}{2}x^2 - 5\frac{x^3}{3} + A\right] \qquad \ldots (4)$$

and $$v = -\frac{1}{EI}\left[\frac{25}{6}x^3 - 5\frac{x^4}{12} + Ax + B\right] \qquad \ldots (5)$$

The *EI* term is taken outside of the integration as both *E* and *I* are constant values. The terms *A* and *B* are *constants of integration.* To solve for the deflection *v* is necessary to solve for *A* and *B* by applying appropriate *boundary conditions* which are (a) when *x*=0 then *v*= 0 and (b) when *x*= 5m then *v*= 0. In other words because the left and right hand ends are both supports then they can not deflect downwards.

A further boundary condition can be deduced in that at mid-span, by symmetry of the beam and loading, the *rotation* (or slope of the curve) which is the term *dv/dx* must be zero. i.e when *x=L/2 = 2.5m* then *dv/dx=0*. The substitution of any two of these three boundary conditions will give *B=0* and *A* = 52.08.

Hence:

$$v = -\frac{1}{EI}\left[\frac{25}{6}x^3 - 5\frac{x^4}{12} + Ax + B\right] = -\frac{1}{EI}\left[\frac{25}{6}x^3 - 5\frac{x^4}{12} + 52.08\right] \qquad \ldots (6)$$

The above expression can now be used to calculate the deflection at any point on the beam. In practice it is the maximum deflection that is of interest and common sense would say that for this example this occurs at mid-span and can be calculated by substituting *x=L/2=2.5m* into equation (6) above.

If it is not obvious where the maximum deflection occurs it can always be determined by knowing that it will occur where there is a change in slope of the beam i.e. where *dv/dx=0* . Hence equation (6), or its equivalent in a similar but different problem, could be differentiated and equated to zero find the distance x_{max} where the rotation *dv/dx* is zero. Substituting this value for x_{max} into equation (6) (or its equivalent) will give the maximum deflection, v_{max}.

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Questions

Example Data: For the steel beams given in figures 6 to 9 check for the following data

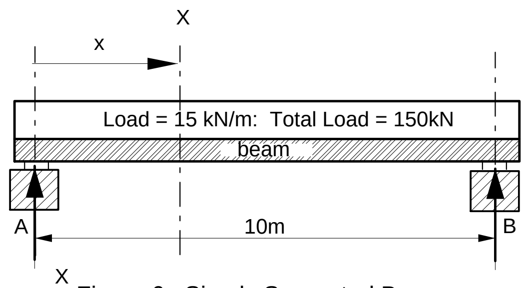

Figure 6: Simply Supported Beam

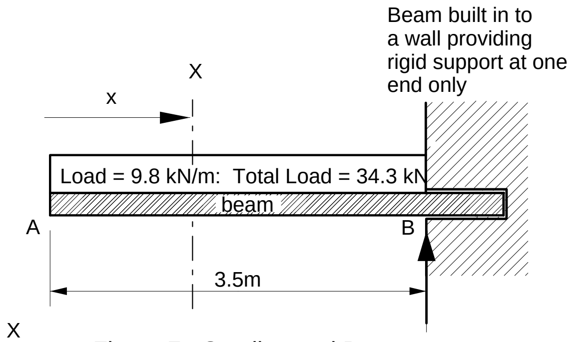

Figure 7: Cantilevered Beam

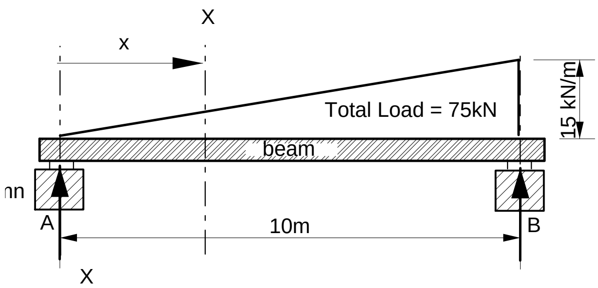

Figure 8: Simply Supported Beam - Tapered Load

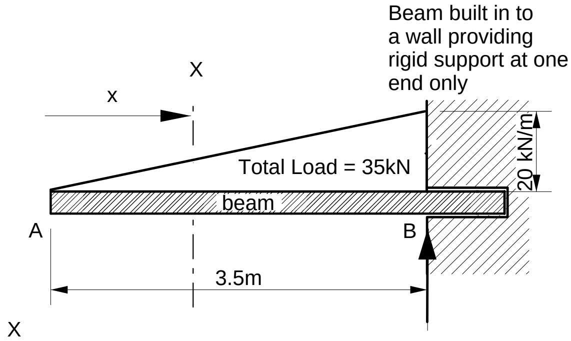

Figure 9: Cantilevered Beam - Tapered Load

Figure	***E*** **(*kN/mm*^{2}*)***	***I*** **(*cm*^{4}*)***	***L*** **(*m*)**	**Load** **(*kN/m*)**	**Calculate deflections at:**
6	210	45,730	10.0	15.0	Mid span of beam
7	210	33,300	3.5	9.8	End of cantilever
8	210	45,730	10.0	Zero to 15 (see figure 8)	Mid span of beam
9	210	37,050	3.5	Zero to 20 (see figure 9)	End of cantilever

Note that two of the problems are based on *cantilever* beams where the beam is held rigidly at one end and is unsupported at the other end. The boundary conditions in this case are that at the built-in end both rotation and deflection will be zero. In all cases, when calculating the equation of Bending Moment, *M*, take moments of all forces to the *left* of the section X-X as shown in the figures.

Where to find more

1. Ray Hulse & Jack Cain, *Structural Mechanics,* 2nd edn, Palgrave, 2000. (ISBN 0-333-80457-0)
2. John Bird, *Engineering Mathematics,* 5th edn, John Bird, 2007 (ISBN 978-07506-8555-9)

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The Mathematics of Beam Deflection

INFORMATION FOR TEACHERS

Teachers will need to understand and explain the theory outlined above and have knowledge of:

- Some terminology relating to structural design and construction
- Integration techniques
- Geometry of the triangle including area and centroid position

Topics covered from Mathematics for Engineers

- Topic 1: Mathematical Models in Engineering
- Topic 2: Models of Growth and Decay
- Topic 6: Differentiation and Integration

Learning Outcomes

- LO 01: Understand the idea of mathematical modelling
- LO 02: Be familiar with a range of models of change, and growth and decay
- LO 06: Know how to use differentiation and integration in the context of engineering analysis and problem solving
- LO 09: Construct rigorous mathematical arguments and proofs in engineering context
- LO 10: Comprehend translations of common realistic engineering contexts into mathematics

Assessment Criteria

- AC 1.1: State assumptions made in establishing a specific mathematical model
- AC 1.2: Describe and use the modelling cycle
- AC 2.3: Set up and solve a differential equation to model a physical situation
- AC 6.3: Find definite and indefinite integrals of functions
- AC 8.3: Use methods of probability to help solve engineering problems
- AC 9.1: Use precise statements, logical deduction and inference
- AC 9.2: Manipulate mathematical expressions
- AC 9.3: Construct extended arguments to handle substantial problems
- AC 10.1: Read critically and comprehend longer mathematical arguments or examples of applications

Links to other units of the Advanced Diploma in Construction & The Built Environment

Unit 3 Civil Engineering Construction
Unit 29 Science and materials in construction and the Built Environment
Unit 30 Structural Mechanics
Unit 31 Design

Solution to the Questions

Figure	Answer Deflection (mm)
6	20.34
7	2.63
8	10.16
9	1.29

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